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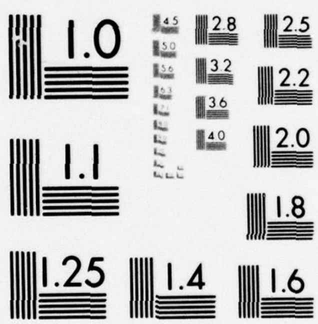

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IDENTITIES FROM PARTITION INVOLUTIONS

by

Donald E. Knuth and Michael S. Paterson

STAN-CS-77-590
FEBRUARY 1977

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Subbarao and Andrews have observed that the combinatorial technique used by F. Franklin to prove Euler's famous partition identity

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots$$

can be applied to prove the more general formula

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$$1-x-x^2y(1-xy)-x^3y^2(1-xy)(1-x^2y)-x^4y^3(1-xy)(1-x^2y)(1-x^3y)-\ldots$$

$$= 1-x-x^2y+x^5y^3+x^7y^4-x^{12}y^6-x^{15}y^7+\ldots$$

which reduces to Euler's when $y = 1$. This note shows that several finite versions of Euler's identity can also be demonstrated using this elementary technique; for example,

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}$$

$$= (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$$

$$-x^7(1-x^2)(1-x^3)(1-x^4)(1-x^5)+x^{7+6}(1-x^3)(1-x^4)-x^{7+6+5}$$

$$= (1-x)(1-x^2)(1-x^3)-x^4(1-x^2)(1-x^3)+x^{4+5}(1-x^3)-x^{4+5+6} .$$

By using Sylvester's modification *of Franklin's construction*, it is also possible to generalize Jacobi's triple product identity.

Identities from Partition Involutions

by

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To George Pólya on the 2^{15}th day after his birth: August 31, 1977.

Abstract.

Subbarao and Andrews have observed that the combinatorial technique used by F. Franklin to prove Euler's famous partition identity

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots \;=\; 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots$$

can be applied to prove the more general formula

$$\begin{aligned}&1-x-x^2y(1-xy)-x^3y^2(1-xy)(1-x^2y)-x^4y^3(1-xy)(1-x^2y)(1-x^3y)-\ldots\\ &\quad= 1-x-x^2y+x^5y^3+x^7y^4-x^{12}y^6-x^{15}y^7+\ldots\end{aligned}$$

which reduces to Euler's when $y=1$. This note shows that several finite versions of Euler's identity can also be demonstrated using this elementary technique; for example,

$$\begin{aligned}&1-x-x^2+x^5+x^7-x^{12}-x^{15}\\ &\quad= (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\\ &\qquad\quad -x^7(1-x^2)(1-x^3)(1-x^4)(1-x^5)+x^{7+6}(1-x^3)(1-x^4)-x^{7+6+5}\\ &\quad= (1-x)(1-x^2)(1-x^3)-x^4(1-x^2)(1-x^3)+x^{4+5}(1-x^3)-x^{4+5+6}\;.\end{aligned}$$

By using Sylvester's modification of Franklin's construction, it is also possible to generalize Jacobi's triple product identity.

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0. Introduction.

Nearly a century ago [7], [14, §12], a young man named Fabian Franklin published what was to become one of the first noteworthy American contributions to mathematics, an elementary combinatorial proof of Euler's well-known identity

$$\prod_{j\geq 1}(1-x^j) = 1-x-x^2+x^5+x^7-\ldots = \sum_{-\infty<k<\infty}(-1)^k x^{(3k^2+k)/2} . \tag{0.1}$$

His approach was to find a nearly one-to-one correspondence between partitions with an even number of distinct parts and those with an odd number of distinct parts, thereby showing that most of the terms on the left-hand side of (0.1) cancel in pairs. Such combinatorial proofs of identities often yield further information, and in the first part of this note we shall demonstrate that Franklin's construction can be used to prove somewhat more than (0.1).

In the second part of this note, we show that Sylvester's modification of Franklin's construction can be applied in a similar way to obtain generalizations of Jacobi's triple product identity

$$\prod_{j\geq 1}(1-q^{2j-1}z)(1-q^{2j-1}z^{-1})(1-q^{2j})$$
$$= 1-q(z+z^{-1})+q^4(z^2+z^{-2})-\ldots = \sum_{-\infty<k<\infty}(-1)^k q^{k^2} z^k . \tag{0.2}$$

1. The Basic Involution.

First let us recall the details of Franklin's construction. Let π be a partition of n into m distinct parts, so that $\pi = \{a_1,\ldots,a_m\}$ for some integers $a_1 > \ldots > a_m > 0$, where $a_1+\ldots+a_m = n$. We shall write

$$\Sigma(\pi) = n \quad , \quad \nu(\pi) = m \quad , \quad \lambda(\pi) = a_1 \quad , \tag{1.1}$$

for the sum, number of parts, and largest part of π, respectively; if π is the empty set, we let $\Sigma(\pi) = \nu(\pi) = \lambda(\pi) = 0$. Following Hardy and Wright [8], we also define the "base" $b(\pi)$ and "slope" $s(\pi)$ as follows:

$$\beta(\pi) = \min\{j \mid j\in\pi\} \quad , \quad \sigma(\pi) = \min\{j \mid \lambda(\pi)-j \notin \pi\} \quad . \tag{1.2}$$

Note that if π is nonempty we have

$$\lambda(\pi) \geq \beta(\pi)+\nu(\pi)-1 \quad \text{and} \quad \nu(\pi) \geq \sigma(\pi) \quad . \tag{1.3}$$

The partition $F(\pi)$ corresponding to π under Franklin's transformation is obtained as follows:

(i) If $\beta(\pi) \leq \sigma(\pi)$ and $\beta(\pi) < \nu(\pi)$, remove the smallest part, $\beta(\pi)$, and increase each of the largest $\beta(\pi)$ parts by one.

(ii) If $\beta(\pi) > \sigma(\pi)$ and $\sigma(\pi) < \nu(\pi)$ or $\sigma(\pi) \neq \beta(\pi)-1$, decrease each of the largest $\sigma(\pi)$ parts by one and append a new smallest part, $\sigma(\pi)$.

(iii) Otherwise $F(\pi) = \pi$. (This case holds if and only if π is empty or $\sigma(\pi) = \nu(\pi) \leq \beta(\pi) \leq \sigma(\pi)+1$.)

These definitions are easily understood in terms of the "Ferrers graph" [14, p. 253] for the partition π, as shown in Figure 1. It is not difficult to verify that F is an involution, i.e., that

$$F(F(\pi)) = \pi \tag{1.4}$$

for all π.

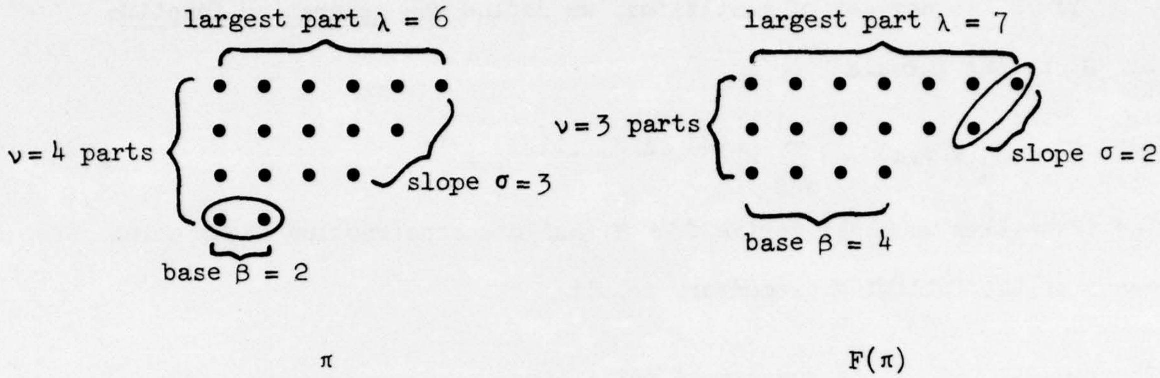

Figure 1. Two partitions of 17 into distinct parts, obtained from each other by moving the two circled elements.

For each $\ell \geq 0$ there is exactly one partition π such that $\lambda(\pi) = \ell$ and $F(\pi) = \pi$. We shall denote this fixed point of the mapping by f_ℓ; it has $\lceil \ell/2 \rceil$ consecutive parts,

$$f_\ell = \{\ell, \ell-1, \ldots, \lfloor \ell/2 \rfloor + 1\} . \tag{1.5}$$

(See Figure 2.) Let

$$\Phi = \{f_0, f_1, f_2, \ldots\} \tag{1.6}$$

be the set of all such partitions. Note that the somewhat similar partitions $\{2k+1, 2k, \ldots, k+2\}$ and $\{2k, 2k-1, \ldots, k\}$ are not fixed under F, although their bases and slopes do intersect.

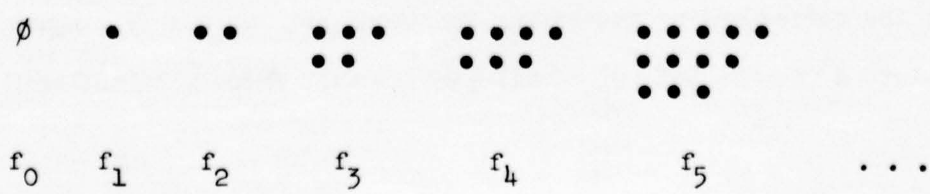

Figure 2. The partitions which remain fixed under F.

2. Extended Generating Functions.

If S is any set of partitions, we define the <u>generating function</u> of S by the formula

$$G_S(x,y,z) = \sum_{\pi \in S} x^{\Sigma(\pi)} y^{\lambda(\pi)} z^{\nu(\pi)} \quad . \tag{2.1}$$

The identities we shall derive from Franklin's construction are special cases of the following elementary result:

Theorem 1. If S is any set of partitions,

$$G_S(x,y,-y) = G_{S \cap \Phi}(x,y,-y) + G_{S \setminus F(S)}(x,y,-y) \quad . \tag{2.2}$$

Proof.

Let π be a partition with $\pi' = F(\pi) \neq \pi$. Then $\Sigma(\pi') = \Sigma(\pi)$, $\lambda(\pi') = \lambda(\pi) \pm 1$, and $\nu(\pi') = \nu(\pi) \mp 1$, hence

$$x^{\Sigma(\pi)} y^{\lambda(\pi)} (-y)^{\nu(\pi)} + x^{\Sigma(\pi')} y^{\lambda(\pi')} (-y)^{\nu(\pi')} = 0 \quad . \tag{2.3}$$

This equation means that π and π' do not contribute to $G_S(x,y,-y)$ if they are both members of S . The only terms which fail to cancel out are from partitions $\pi \in S$ with $F(\pi) = \pi$, namely the elements of $S \cap \Phi$, and those from partitions $\pi \in S$ with $F(\pi) \notin S$, namely the elements of $S \setminus F(S)$. $\square$

3. Three Identities.

In order to get interesting corollaries of Theorem 1, we must find sets S for which the corresponding generating functions are reasonably simple.

First, let S be the set P of all partitions. Theorem 1 implies that

$$G_P(x,y,-y) = G_\Phi(x,y,-y) \quad . \tag{3.1}$$

Now

$$G_P(x,y,z) = 1 + \sum_{\ell \geq 1} x^\ell y^\ell z \prod_{1 \leq j < \ell} (1 + x^j z) \tag{3.2}$$

and

$$G_\Phi(x,y,z) = 1 + \sum_{\ell \geq 1} x^{\ell(\ell+1)/2 - \lfloor \ell/2 \rfloor (\lfloor \ell/2 \rfloor + 1)/2} y^\ell z^{\lceil \ell/2 \rceil} \tag{3.3}$$

$$= 1 + \sum_{k \geq 1} \left(x^{(3k^2-k)/2} y^{2k-1} z^k + x^{(3k^2+k)/2} y^{2k} z^k \right) . \tag{3.4}$$

Thus we have

<u>Corollary 1.1.</u>

$$\sum_{\ell \geq 1} x^\ell y^{\ell+1} \prod_{1 \leq j < \ell} (1 - x^j y) = \sum_{k \geq 1} (-1)^{k-1} \left(x^{(3k^2-k)/2} y^{3k-1} + x^{(3k^2+k)/2} y^{3k} \right) . \tag{3.5}$$

Franklin essentially considered the special case $y = 1$ of this identity, when the left-hand side reduces to $1 - \prod_{j \geq 1} (1 - x^j)$. Equation (3.5) was originally discovered by L. J. Rogers [10, §10(4)], who gave an analytic proof. The fact that Franklin's correspondence could be used to obtain (3.5) was first noticed by M. V. Subbarao [12] and G. E. Andrews [2].

Although the power series identity of Corollary 1.1 is formally true, it does not converge for all x and y; for example, if we set $y = x^{-1}$ we get the anomalous formula $x^{-1} = x^{-1} + x^{-1} - 1 - x + x^4 + x^6 - \ldots$. To better understand the rate of convergence, we can obtain an exact truncated version of the sum by restricting S to the set

$$P_n = \{\lambda(\pi) \leq n\} \quad . \tag{3.6}$$

Since

$$\begin{aligned} P_n \backslash F(P_n) &= \{\pi \mid \lambda(\pi) = n \text{ and } \beta(\pi) \leq \sigma(\pi) \text{ and } \beta(\pi) < \nu(\pi)\} \\ &= \{\pi \mid \lambda(\pi) = n \text{ and } \beta(\pi) \leq \sigma(\pi) \text{ and } \beta(\pi) \leq n/2\} \end{aligned} \tag{3.7}$$

we have

$$G_{P_n \setminus F(P_n)}(x,y,z) = \sum_{1 \le b \le n/2} (x^b y^n z) \left(\prod_{b < j \le n-b} (1+x^j z) \right) \left(\prod_{n-b < j \le n} x^j z \right) . \quad (3.8)$$

Thus Theorem 1 yields

Corollary 1.2.

$$\sum_{1 \le \ell \le n} x^\ell y^{\ell+1} \prod_{1 \le j < \ell} (1-x^j y) = \sum_{1 \le k \le (n+1)/2} (-1)^{k-1} x^{(3k^2-k)/2} y^{3k-1}$$

$$+ \sum_{1 \le k \le n/2} (-1)^{k-1} x^{(3k^2+k)/2} y^{3k}$$

$$+ \sum_{1 \le b \le n/2} (-1)^b y^{n+b+1} \left(\prod_{b < j \le n-b} (1-x^j y) \right) \left(\prod_{n-b < j \le n} x^{j+1} \right) .$$

For example, the cases $n = 4$ and $n = 5$ of this identity are

$$xy^2 + x^2y^3(1-xy) + x^3y^4(1-xy)(1-x^2y) + x^4y^5(1-xy)(1-x^2y)(1-x^3y)$$
$$= xy^2 + x^2y^3 - x^5y^5 - x^7y^6 - x^5y^6(1-x^2y)(1-x^3y) + x^{5+4}y^7 \quad ; \quad (3.9)$$

$$xy^2 + x^2y^3(1-xy) + x^3y^4(1-xy)(1-x^2y) + x^4y^5(1-xy)(1-x^2y)(1-x^3y)$$
$$+ x^5y^6(1-xy)(1-x^2y)(1-x^3y)(1-x^4y)$$
$$= xy^2 + x^2y^3 - x^5y^5 - x^7y^6 + x^{12}y^8 - x^6y^7(1-x^2y)(1-x^3y)(1-x^4y) + x^{6+5}y^8(1-x^3y) . \quad (3.10)$$

Setting $y = 1$ and subtracting both sides from 1 yields truncated versions of Euler's formula which appear to be new; e.g.,

$$1 - x - x^2 + x^5 + x^7 = (1-x)(1-x^2)(1-x^3)(1-x^4) - x^5(1-x^2)(1-x^3) + x^{5+4} \quad ; \quad (3.11)$$

$$1-x-x^2+x^5+x^7-x^{12} = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)$$

$$-x^6(1-x^2)(1-x^3)(1-x^4)+x^{6+5}(1-x^3) \quad ; \qquad (3.12)$$

$$1-x-x^2+x^5+x^7-x^{12}-x^{15} = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$$

$$-x^7(1-x^2)(1-x^3)(1-x^4)(1-x^5)+x^{7+6}(1-x^3)(1-x^4)-x^{7+6+5} \quad . \qquad (3.13)$$

Essentially the same formulas, but with n decreased by 2, would have been obtained if we had set $y = x^{-1}$ in the identity of Corollary 1.2.

Let us also consider another family of partition sets with a reasonably simple generating function,

$$S_n = \{\pi \mid \beta(\pi) > \lambda(\pi)-n \text{ and } \sigma(\pi) \geq \lambda(\pi)-n\} \quad . \qquad (3.14)$$

These sets are closed under F, for if $\pi' = F(\pi) \neq \pi$ we have either (i) $\lambda(\pi') = \lambda(\pi)+1$, $\beta(\pi') \geq \beta(\pi)+1$, and $\sigma(\pi') = \beta(\pi)$, or (ii) $\lambda(\pi') = \lambda(\pi)-1$, $\beta(\pi') \geq \sigma(\pi)$, and $\sigma(\pi') \geq \sigma(\pi)$. Note that S_n is finite, since $\pi \in S_n$ implies that $2\lambda(\pi)-2n \leq \beta(\pi)+\sigma(\pi)-1 \leq \lambda(\pi)$, hence $\lambda(\pi) \leq 2n$. The set of fixed points $S_n \cap \Phi$ is $\{f_0, f_1, \ldots, f_{2n}\}$, and

$$G_{S_n}(x,y,z) = G_{P_n}(x,y,z) + \sum_{n<\ell\leq 2n} x^\ell y^\ell z \left(\prod_{\ell-n<j\leq n}(1+x^j z)\right)\left(\prod_{n<j<\ell} x^j z\right), \qquad (3.15)$$

so Theorem 1 yields a companion to Corollary 1.2:

Corollary 1.3.

$$\sum_{1 \le \ell \le n} x^{\ell} y^{\ell+1} \prod_{1 \le j < \ell} (1-x^j y) = \sum_{1 \le k \le n} (-1)^{k-1} \left(x^{(3k^2-k)/2} y^{3k-1} + x^{(3k^2+k)/2} y^{3k} \right)$$

$$+ \sum_{1 \le b \le n} (-1)^b y^{2b+n} \left(\prod_{b < j \le n} (1-x^j y) \right) \left(\prod_{n < j \le n+b} x^j \right) .$$

For example, the cases $n = 2, 3$ of this identity are

$$xy^2 + x^2y^3(1-xy) = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 - x^3y^4(1-x^2y) + x^{3+4}y^6 \quad ;$$

$$xy^2 + x^2y^3(1-xy) + x^3y^4(1-xy)(1-x^2y) = xy^2 + x^2y^3 - x^5y^5 - x^7y^6 + x^{12}y^8 + x^{15}y^9$$

$$- x^4y^5(1-x^2y)(1-x^3y) + x^{4+5}y^7(1-x^3y) - x^{4+5+6}y^9 \quad .$$

Setting $y = 1$ and subtracting from 1 leads to formulas somewhat analogous to (3.11) and (3.13):

$$1 - x - x^2 + x^5 + x^7 = (1-x)(1-x^2) - x^3(1-x^2) + x^{3+4} \quad ; \tag{3.16}$$

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} = (1-x)(1-x^2)(1-x^3) - x^4(1-x^2)(1-x^3)$$
$$+ x^{4+5}(1-x^3) - x^{4+5+6} \quad . \tag{3.17}$$

Let us restate the identities arising from Corollaries 1.2 and 1.3 when $y = 1$, where n is even in Corollary 1.2:

$$1+\sum_{1\le k\le n}(-1)^k\, x^{(3k^2-k)/2}+x^{(3k^2+k)/2}$$

$$=\sum_{0\le k\le n}(-1)^k\, x^{(2n+2)k-k(k+1)/2}\prod_{k<j\le 2n-k}(1-x^j) \tag{3.18}$$

$$=\sum_{0\le k\le n}(-1)^k\, x^{nk+k(k+1)/2}\prod_{k<j\le n}(1-x^j)\quad . \tag{3.19}$$

The latter formula was discovered by D. Shanks [11] in the course of some experiments on nonlinear transformations of series; he observed that it can be proved by induction on n without great difficulty. There is also a short proof of (3.18): Let

$$A(k,n) = (1-x^k)+x^k(1-x^k)(1-x^{k+1})+\ldots+x^{kn}(1-x^k)\ldots(1-x^{k+n})\quad , \tag{3.20}$$

$$R(k,n) = x^{(n+1)k}(1-x^{k+1})\ldots(1-x^{k+n})\quad . \tag{3.21}$$

Then $A(0,n)=0$, $A(k,0)=1-x^k$, $A(k,-1)=0$, and it is not difficult to show that

$$A(k,n) = 1-x^{2k+1}-R(k,n)-x^{3k+2}A(k+1,n-2)\qquad \text{if}\quad n>0\ . \tag{3.22}$$

Iteration of this recurrence yields identity (3.18). The use of this recurrence is actually only a slight extension of Euler's original technique [6] for proving (0.1).

It is interesting to compare (3.18) and (3.19) to "classical" formulas on terminating basic hypergeometric series, as suggested in a note to the authors by G. E. Andrews. If we set $a=1$, $b=c=d=\infty$, and $q=x$ in a highly general identity given by R. P. Agarwal [1, Eq. (4.2)], we obtain

$$1 + \sum_{1 \le k \le n} (-1)^k \, x^{(3k^2-k)/2} + x^{(3k^2+k)/2}$$

$$= \sum_{0 \le k \le n} (-1)^k \, x^{k(k+1)/2} \left(\prod_{k < j \le 2n-k} (1-x^j) \right) \Big/ \prod_{1 \le j \le n-k} (1-x^j) \; . \qquad (3.23)$$

In particular, when $n = 3$ this formula gives the following analog of (3.13) and (3.17):

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)}{(1-x)(1-x^2)(1-x^3)}$$

$$- x^1 \frac{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}{(1-x)(1-x^2)} + x^{1+2} \frac{(1-x^3)(1-x^4)}{(1-x)} - x^{1+2+3} \; . \qquad (3.24)$$

4. Sylvester's Involution.

Let us now turn to Jacobi's identity (0.2), which is formally equivalent under the substitution $q^2 = uv$ and $z^2 = uv^{-1}$ to

$$\prod_{j\geq 1}(1-u^jv^{j-1})(1-u^jv^j)(1-u^{j-1}v^j)$$

$$= 1+\sum_{k\geq 1}(-1)^k\left(u^{(k^2+k)/2}v^{(k^2-k)/2}+u^{(k^2-k)/2}v^{(k^2+k)/2}\right) . \tag{4.1}$$

The left-hand side of this equation can be interpreted as involving partitions of Gaussian integers $m+ni$ into distinct parts of the form $p+qi$, where $\max(p,q) > 0$ and $|p-q| \leq 1$; the coefficient of $u^m v^n$ will be the excess of the number of such partitions with an even number of parts over those with an odd number of parts. The right-hand side says that there exists a nearly one-to-one correspondence between such even and odd partitions, the only unmatched partitions being of the forms

$$\{1,2+i,\ldots,k+(k-1)i\} \quad\text{or}\quad \{i,1+2i,\ldots,k-1+ki\} . \tag{4.2}$$

An explicit correspondence of this sort was discovered by J. J. Sylvester [14, §§57-61, 64-68] shortly after he had learned of Franklin's construction; at that time Sylvester was a professor at Johns Hopkins University in Baltimore.*/

*/ The literature contains several incorrect references to the history of Sylvester's construction. Sudler [13] says that the approach taken by Wright [15] is essentially that of Sylvester; but in fact it is essentially the same as another construction due to Arthur S. Hathway, quoted by Sylvester in [14, §62]. Zolnowsky [16] independently rediscovered Sylvester's rules (i) - (iv), and observed that these were sufficient to prove Jacobi's identity since they will handle all cases $m+ni$ with $m \geq n$.

Sylvester's original treatment has apparently never been cited by anyone else, possibly because it comes at the end of a very long paper; furthermore his notation was rather obscure, and he made numerous careless errors that a puzzled reader must rectify. Indeed, the present authors may never have been able to understand what Sylvester was talking about if Zolnowsky's clear presentation had not been available.

We shall represent complex partitions π by three real partitions π_+ , π_0 , π_- , containing respectively $\max(p,q)$ for those parts $p+qi$ in which $p-q = +1$, 0 , or -1 . For example, the complex partition

$$\pi = \{3+2i\ ,\ 2+i\ ,\ 1\ ,\ 3+3i\ ,\ 1+i\ ,\ 3+4i\}$$

of $13+11i$ will be represented by

$$\pi_+ = \{3,2,1\}\ ,\quad \pi_0 = \{3,1\}\ ,\quad \pi_- = \{4\}\ .$$

Sylvester noted that if i is artificially set equal to 2 , we obtain a one-to-one correspondence between the complex partitions of $m+ni$ and a subset of the real partitions of $m+2n$ into distinct parts; π_+ , π_0 , and π_- map into the parts congruent respectively to $+1$, 0 , and -1 modulo 3 , hence Jacobi's identity implies Euler's.

In order to present Sylvester's construction, we recall the definitions of $\Sigma(\pi)$, $\nu(\pi)$, $\lambda(\pi)$, $\beta(\pi)$, and $\sigma(\pi)$ for real partitions in Section 1 above; we also add two more attributes,

$$\tau[\pi] = \min\{k \mid k+1 \notin \pi\}\ , \tag{4.3}$$

$$\alpha[\ell] = \min\{k \mid k \in \pi \text{ and } k > \tau(\pi)\}\ . \tag{4.4}$$

By convention, the minimum over an empty set is ∞ ; thus, $\beta[\pi] = \infty$ if and only if π is empty, and $\alpha[\pi] = \infty$ if and only if π has the form $\{1,2,\ldots,k\}$ for some $k \geq 0$. Sylvester defined an involution $F(\pi)$ on complex partitions π by what amounts to the following seven rules:

(i) If $\beta(\pi_0) \leq \sigma(\pi_+)$, remove the smallest part, $\beta(\pi_0)$, from π_0 and increase each of the largest $\beta(\pi_0)$ parts of π_+ by one.

(ii) If $\beta(\pi_0) > \sigma(\pi_+) > 0$ and $\sigma(\pi_+) \neq \lambda(\pi_+)$, decrease each of the largest $\sigma(\pi_+)$ parts of π_+ by one and append a new smallest part, $\sigma(\pi_+)$, to π_0 .

(iii) If $\beta(\pi_0) > \sigma(\pi_+) = \lambda(\pi_+)$ and $\beta(\pi_0) < \sigma(\pi_+) + \beta(\pi_-)$, remove the smallest part, $\beta(\pi_0)$, from π_0 and append a new largest part, $\sigma(\pi_+)+1$, to π_+ and a new smallest part, $\beta(\pi_0) - \sigma(\pi_+)$, to π_-.

(iv) If $\beta(\pi_0) > \sigma(\pi_+) = \lambda(\pi_+) > 0$ and $\beta(\pi_0)+1 > \sigma(\pi_+) + \beta(\pi_-)$, remove the largest part, $\sigma(\pi_+)$, from π_+ and the smallest part, $\beta(\pi_-)$, from π_- and append a new smallest part, $\sigma(\pi_+) + \beta(\pi_-)-1$, to π_0.

(v) If $\lambda(\pi_+) = 0$ and $\alpha(\pi_-) > \beta(\pi_0) + \tau(\pi_-)$ and $\tau(\pi_-) > 0$, remove the smallest part, $\beta(\pi_0)$, from π_0 and replace the part $\tau(\pi_-)$ in π_- by $\tau(\pi_-) + \beta(\pi_0)$.

(vi) If $\lambda(\pi_+) = 0$ and $\alpha(\pi_-) < \beta(\pi_0) + \tau(\pi_-)+1$, replace the part $\alpha(\pi_-)$ in π_- by $\tau(\pi_-)+1$, and append a new smallest part, $\alpha(\pi_-) - \tau(\pi_-)-1$, to π_0.

(vii) Otherwise $F(\pi) = \pi$. (This happens if and only if π has the form (4.2).)

It can be shown that $F(F(\pi)) = \pi$, and that in fact rules (i) - (ii), (iii) - (iv), (v) - (vi) undo each other.*/

For example, Sylvester's correspondence pairs up the complex partitions in the following way, if we denote partitions by listing the respective elements of π_+, π_0, π_- separated by vertical bars**/:

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*/ At this point one cannot resist quoting Sylvester, who stated that these rules possess what he called Catholicity, Homoeogenesis, Mutuality, Inertia, and Enantiotropy: "I need hardly say that so highly organized a scheme ... has not issued from the mind of its composer in a single gush, but is the result of an analytical process of continued residuation or successive heaping of exception upon exception in a manner dictated at each point in its development by the nature of the process and the resistance, so to say, of its subject-matter." [14, p. 314]

**/ These are the complex partitions whose sums have the form $k+(11-2k)i$. Sylvester gave an incorrect table corresponding to these 12 partitions at the bottom of [14, p. 315]; in his notation, he should have written

"1st Species. 11 3.8; 6.3.2 6.5; 8.2.1 3.5.2.1.
2d Species. 9.2 5.2.4.
3d Species. 10.1 6.4.1; 7.4 3.7.1."

$3|1| \quad \leftrightarrow \quad 4||$ rules (i) and (ii)

$21|1|1 \quad \leftrightarrow \quad 31||1$ rules (i) and (ii)

$1|21| \quad \leftrightarrow \quad 2|2|$ rules (i) and (ii)

$1|3| \quad \leftrightarrow \quad 21||2$ rules (iii) and (iv)

$|2|21 \quad \leftrightarrow \quad ||41$ rules (v) and (vi)

$|1|31 \quad \leftrightarrow \quad ||32$ rules (v) and (vi)

5. Generating Functions Revisited.

If S is a set of complex partitions, we let

$$G_S(u,v,y,z) = \sum_{\pi \in S} u^{\mathcal{R}\Sigma(\pi)} v^{\mathcal{I}\Sigma(\pi)} y^{\lambda(\pi)} z^{\nu(\pi_0)} , \tag{5.1}$$

where

$$\mathcal{R}\Sigma(\pi) = \Sigma(\pi_+) + \Sigma(\pi_0) + \Sigma(\pi_-) - \nu(\pi_-) \quad ;$$

$$\mathcal{I}\Sigma(\pi) = \Sigma(\pi_+) - \nu(\pi_+) + \Sigma(\pi_0) + \Sigma(\pi_-) \quad ;$$

$$\lambda(\pi) = \begin{cases} \lambda(\pi_+) & \text{if } \lambda(\pi_+) > 0 \;; \\ -\tau(\pi_-) & \text{if } \lambda(\pi_+) = 0 \;. \end{cases} \tag{5.2}$$

These definitions have the property we want, as shown in the following theorem.

<u>Theorem 2</u>. Let S be any set of complex partitions, and let Φ be the set of all complex partitions of the form (4.2). Then

$$G_S(u,v,y,-y) = G_{S \cap \Phi}(u,v,y,-y) + G_{S \setminus F(S)}(u,v,y,-y) \; . \tag{5.3}$$

Proof. As in Theorem 1, we need only verify that if $\pi' = F(\pi) \neq \pi$ we have $\Sigma(\pi') = \Sigma(\pi)$, $\lambda(\pi') = \lambda(\pi) \pm 1$, and $\nu(\pi_0') = \nu(\pi_0) \mp 1$. Rules (i), (iii), (v) all leave Σ unchanged, decrease $\nu(\pi_0)$, and increase $\lambda(\pi)$; rules (ii), (iv), (vi) are the inverses. There is one slightly subtle case worth discussing: Rule (iii) applies when $\lambda(\pi_+) = 0$ and it changes $\lambda(\pi_+)$ to 1; in that case the hypothesis $\beta(\pi_0) < \beta(\pi_-)$ implies that $\tau(\pi_-) = 0$, hence $\lambda(\pi) = 0$. □

6. Jacobi-like Identities.

We shall apply Theorem 2 only to two infinite sets of partitions, leaving it to the reader to discover interesting finite versions of Jacobi's identity analogous to Corollaries 1.2 and 1.3.

If P is the set of all complex partitions, we have

$$G_P(u,v,y,z) = \left(\sum_{\ell \geq 1} u^\ell v^{\ell-1} y^\ell \left(\prod_{1 \leq j < \ell} (1+u^j v^{j-1})\right)\left(\prod_{j \geq 1} (1+u^{j-1} v^j)\right)\right.$$
$$\left. + \sum_{\ell \geq 0} y^{-\ell} \left(\prod_{1 \leq j \leq \ell} u^{j-1} v^j\right)\left(\prod_{j > \ell+1} (1+u^{j-1} v^j)\right)\right) \prod_{j \geq 1} (1+u^j v^j z); \quad (6.1)$$

Furthermore

$$G_\Phi(u,v,y,z) = 1 + \sum_{k \geq 1} \left(u^{(k^2+k)/2} v^{(k^2-k)/2} y^k + u^{(k^2-k)/2} v^{(k^2+k)/2} y^{-k}\right). \quad (6.2)$$

Setting $z = -y$ in (6.1) gives the identity $G_P(u,v,y,-y) = G_\Phi(u,v,y,-y)$, which can be rewritten as

Corollary 2.1.

$$\sum_{-\infty<\ell<\infty} \frac{y^\ell u^\ell v^{\ell-1}}{\prod_{j\geq 0}(1+u^{j+\ell}v^{j+\ell-1})} \left(\prod_{j\geq 1} (1+u^{j-1}v^j)(1+u^j v^{j-1})(1-u^j v^j y) \right)$$

$$= \sum_{-\infty<k<\infty} u^{(k^2+k)/2} v^{(k^2-k)/2} y^k \quad .$$

Our derivation makes it clear that this formula reduces to (4.1) if we set $y = 1$ and replace (u,v) by $(-u,-v)$; it is therefore a three-parameter generalization of Jacobi's identity.

The right-hand side of Corollary 2.1 can be expressed as

$$\sum_{-\infty<k<\infty} (uy)^{(k^2+k)/2} (vy^{-1})^{(k^2-k)/2} = \prod_{j\geq 1} (1+u^{j-1}v^j y^{-1})(1+u^j v^{j-1} y)(1-u^j v^j)$$

by Jacobi's identity (4.1), hence Corollary 2.1 implies that

$$\sum_{-\infty<\ell<\infty} \frac{y^\ell u^\ell v^{\ell-1}}{\prod_{j\geq 0}(1+u^{j+\ell}v^{j+\ell-1})}$$

$$= \prod_{j\geq 1} \frac{(1+u^{j-1}v^j y^{-1})}{(1+u^{j-1}v^j)} \frac{(1+u^j v^{j-1} y)}{(1+u^j v^{j-1})} \frac{(1-u^j v^j)}{(1-u^j v^j y)} \quad .$$

Let us set $a = -v^{-1}$, $q = uv$, and $x = uvy$, to make the structure of this formula slightly more clear; we obtain

$$\sum_{-\infty < n < \infty} \frac{x^n}{\prod_{j \geq 0} (1-aq^{j+n})} = \prod_{k \geq 0} \frac{(1-a^{-1}x^{-1}q^{j+1})(1-axq^j)(1-q^{j+1})}{(1-a^{-1}q^{j+1})(1-aq^j)(1-xq^j)} \quad . \tag{6.3}$$

This three-parameter identity turns out to be merely the special case $b = 0$ of a "remarkable formula with many parameters" discovered by S. Ramanujan (see [8, Eq. (12.12.2)]); Ramanujan's formula, for which a surprisingly simple analytic proof has recently been found [5], can be written

$$\sum_{-\infty < n < \infty} x^n \prod_{j \geq 0} \left(\frac{1-bq^{j+n}}{1-aq^{j+n}} \right)$$

$$= \prod_{j \geq 0} \frac{(1-ba^{-1}q^j)(1-a^{-1}x^{-1}q^{j+1})(1-axq^j)(1-q^{j+1})}{(1-ba^{-1}x^{-1}q^j)(1-a^{-1}q^{j+1})(1-aq^j)(1-xq^j)} \quad . \tag{6.4}$$

If we let S be the set of all complex partitions with π_+ nonempty, $G_S(u,v,y,z)$ and $G_{S \cap \Phi}(u,v,y,z)$ are given by the terms in (6.1) and (6.2) involving y^ℓ for $\ell \geq 1$. The set $S \setminus F(S)$ consists of those partitions with $\pi_+ = \{1\}$ and $\beta(\pi_-) < \beta(\pi_0)$, hence

$$G_{S \setminus F(S)}(u,v,y,z) = uy \sum_{b \geq 1} u^{b-1}v^b \prod_{j > b} (1+u^jv^jz)(1+u^{j-1}v^j) \quad .$$

By Theorem 2, we obtain

<u>Corollary 2.2.</u>

$$\left(\sum_{\ell \geq 1} u^\ell v^{\ell-1} y^\ell \prod_{1 \leq j < \ell} (1+u^jv^{j-1}) \right) \left(\prod_{j \geq 1} (1-u^jv^jy)(1+u^{j-1}v^j) \right)$$

$$= \sum_{k \geq 1} u^{(k^2+k)/2} v^{(k^2-k)/2} y^k + y \sum_{b \geq 1} u^b v^b \prod_{j > b} (1-u^jv^jy)(1+u^{j-1}v^j) \quad .$$

If we subtract this identity from that of Corollary 2.1, we get the formula for the complement of S , namely

$$\sum_{\ell \geq 0} y^{-\ell} \left(\prod_{1 \leq j \leq \ell} u^{j-1} v^j \right) \left(\prod_{j > \ell+1} (1+u^{j-1} v^j) \right) \left(\prod_{j \geq 1} (1-u^j v^j y) \right)$$

$$= \sum_{k \geq 0} u^{(k^2-k)/2} v^{(k^2+k)/2} y^{-k} - y \sum_{b \geq 1} u^b v^b \prod_{j > b} (1-u^j v^j y)(1+u^{j-1} v^j) \, . \qquad (6.5)$$

Putting $y = 1$ reduces the left-hand side to $\prod_{j>0}(1-u^j v^j)(1+u^{j-1}v^j)$; hence we obtain

$$\sum_{b \geq 0} u^b v^b \prod_{j > b} (1-u^j v^j)(1+u^{j-1} v^j) = \sum_{k \geq 0} u^{(k^2-k)/2} v^{(k^2+k)/2} \, . \qquad (6.6)$$

Let $q = uv$ and $x = -u^{-1}$; *this formula is equivalent to the identity*

$$\sum_{b \geq 0} q^b \prod_{j > b} (1-q^j)(1-q^j x) = \sum_{k \geq 0} (-x)^k q^{(k^2+k)/2} \, . \qquad (6.7)$$

Equation (6.7) can be derived readily from known identities on basic hypergeometric functions. Let us first divide both sides by $\prod_{j \geq 1}(1-q^j)(1-q^j x)$, obtaining

$$\sum_{n \geq 0} \frac{q^n}{\prod_{0 \leq j < n} (1-xq^{j+1})(1-q^{j+1})}$$

$$= \left(\frac{1}{\prod_{j \geq 0} (1-xq^{j+1})(1-q^{j+1})} \right) \sum_{k \geq 0} (-x)^k q^{(k^2+k)/2} \, .$$

Now we use E. Heine's important transformation of such series, a five-parameter identity [9, Eq. 79] which essentially states that

$f(u,v;a,b;q) = f(v,u;b,a;q)$ if

$$f(u,v;a,b;q) = \left(\sum_{n\geq 0} u^n \prod_{0\leq j<n} \frac{(1-aq^j)(1-vq^j)}{(1-bvq^j)(1-q^{j+1})}\right)\left(\prod_{j\geq 0}\left(\frac{1-uq^j}{1-auq^j}\right)\right) . \tag{6.8}$$

In our case we let $u = q$, $v = x/b$, $a = 0$, and $b \to \infty$, obtaining the desired result:

$$\left(\sum_{n\geq 0} \frac{q^n}{\prod_{0\leq j<n}(1-xq^{j+1})(1-q^{j+1})}\right)\left(\prod_{j\geq 0}(1-q^{j+1})\right)$$

$$= \left(\sum_{n\geq 0} x^n \prod_{0\leq j<n}(-q^j)\right)\left(\prod_{j\geq 0}\frac{1}{(1-xq^{j+1})}\right) .$$

It is not clear whether or not the more general equation (6.5) is related to known formulas in an equally simple way.

An amusing special case of (6.7) can be obtained by setting $q = x^2$ and multiplying both sides by x:

$$\sum_{k \text{ odd}} x^k \prod_{j>k}(1-x^j) = x - x^4 + x^9 - \ldots = \sum_{k\geq 0}(-1)^k x^{(k+1)^2} . \tag{6.9}$$

"The partitions of n into an odd number of distinct parts in which the least part is odd are equinumerous with its partitions into an even number of distinct parts in which the least part is odd, unless n is a perfect square." An equivalent statement was posed as a problem by G. E. Andrews several years ago [3], and he has sketched a combinatorial proof in [4, pp. 156-157]. However, there must be an involution on partitions which proves this formula! If the reader can find one, it might well lead to a number of interesting new identities.

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